MINIMAL FREE RESOLUTIONS OF THE G-PARKING FUNCTION IDEAL AND THE TOPPLING IDEAL

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Abstract. The G-parking function ideal M_G of a directed multigraph G is a monomial ideal which encodes some of the combinatorial information of G. It is an initial ideal of the toppling ideal I_G, a lattice ideal intimately related to the chip-firing game on a graph. Both ideals were first studied by Cori, Rossin, and Salvy. A minimal free resolution for M_G was given by Postnikov and Shaprio in the case when G is saturated, i. e., whenever there is at least one edge (u, v) for every ordered pair of distinct vertices u and v. They also raised the problem of an explicit description of the minimal free resolution in the general case. In this paper, we give a minimal free resolution of M_G for any undirected multigraph G, as well as for a family of related ideals including the toppling ideal I_G. This settles a conjecture of Manjunath and Sturmfels, as well as a conjecture of Perkinson and Wilmes.

1. Introduction

Let G be a directed multigraph on n vertices with labels in $[n]$. (By "multigraph" we mean that every directed edge has a nonnegative integer weight.) The adjacency matrix $A_G = (a_{ij})$ of G has rows and columns indexed by vertices, with a_{ij} the weight of the edge (i, j) if it exists, and zero otherwise. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$. The G-**parking function ideal** is

$$M_G = \langle \mathbf{x}^{S \to \overline{S}} : S \subset [n-1] \rangle \subset R,$$

where

$$\mathbf{x}^{S \to \overline{S}} = \prod_{i \in S} x_i^{\sum_{j \notin S} a_{ij}}.$$

(Note that vertex n, which will be the "sink vertex" of G, never appears in S but always appears in $\overline{S}$ in the above definition.) The ideal was first studied by Cori, Rossin, and Salvy [5] in the case of undirected graphs G, and subsequently in the full generality of directed multigraphs by Postnikov and Shapiro [17]. They gave an explicit minimal free resolution in the case that G is saturated, i. e., when the off-diagonal entries of the adjacency matrix are nonzero. In the same paper, Postnikov and Shapiro asked for an explicit description of the minimal free resolution in the general case. We resolve the question in this paper for undirected multigraphs G.

We also describe the minimal free resolution of a lattice ideal, called the toppling ideal I_G, associated with the graph G. Before defining I_G, let us briefly recall the definition of a lattice ideal. Let $\mathcal{L}$ be a sublattice of $\mathbb{Z}^n$. The lattice ideal $I_{\mathcal{L}}$ over R is the ideal generated by binomials whose exponents differ by a point in $\mathcal{L}$. More precisely,

$$I_{\mathcal{L}} = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \mathbf{u} - \mathbf{v} \in \mathcal{L} \rangle,$$

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where $\mathbf{x}^{\mathbf{w}} = x_1^{w_1} \cdots x_n^{w_n}$ for any $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$.

Lattice ideals are generalizations of toric ideals and many attempts have been made to describe their minimal free resolutions. While explicit descriptions of nonminimal free resolutions are known, an explicit description of the minimal free resolution of a lattice ideal is known only in a few cases [14, Chapter 9]. In fact, the well-studied problem of determining the minimal syzygies of the Veronese embedding of $\mathbb{P}^n$ [10] can be rephrased in terms of minimal free resolutions of lattice ideals.

Toppling ideals, the lattice ideals studied in this paper, are connected to the Laplacian of a graph. If G is a connected undirected multigraph, the Laplacian matrix of G is $\Lambda_G = \Delta - A_G$, where Δ is the diagonal matrix with entries $\Delta_{ii} = \deg(i) = \sum_j a_{ij}$. The **toppling ideal** I_G is the lattice ideal defined by the Laplacian lattice $\mathbb{Z}^n \cdot \Lambda_G$. The G-parking function ideal and toppling ideal are closely related: in particular, M_G is an initial ideal of I_G [5]. An inhomogeneous version of the toppling ideal was studied by Cori, Rossin, and Salvy in [5], and the homogeneous version was subsequently studied by Perkinson, Perlman, and Wilmes in [16], and by Manjunath and Sturmfels in [13].

The family of toppling ideals appears to be quite rich. Indeed, Perkinson, Perlman, and Wilmes [16] show that any lattice ideal defined by a full-rank submodule of the root lattice $A_n = \{\mathbf{x} \in \mathbb{Z}^{n+1} : \sum_{i=1}^{n+1} x_i = 0\}$ is the toppling ideal of some *directed* multigraph. No effective characterizations of Laplacian lattices of undirected multigraphs are known, though certainly not every full-rank sublattice of A_n is of this form [1, 16]. Nevertheless, even in the undirected case examined in this paper, the ideals I_G and M_G represent broad, natural categories of ideals. They are closely tied to the graph chip-firing game, or abelian sandpile model, first described by Dhar [6]. The ideal I_G is the lattice ideal corresponding to the lattice of principal divisors of G, and carries information about the Riemann-Roch theory of G (see [13, 2]). The study of its minimal free resolution is natural in this context.

In this paper, we will provide an explicit description of the minimal free resolution of the ideals M_G and I_G for connected undirected multigraphs G. We verify a conjecture made in [13, Conjecture 29] that the Betti numbers of I_G and M_G coincide (Theorem 1.1). Our construction of the minimal free resolution of I_G also proves [20, Conjecture 3.28], which describes how combinatorial information of the graph is encoded in the minimal free resolution. A weaker form of the conjecture, stated in terms of Betti numbers, appeared in [16, Conjecture 7.9], and also follows from Theorem 1.1. Furthermore, as we show in the final section of the paper, the minimal free resolution of M_G is supported by a CW-complex. The resolution for M_G is a Koszul complex when G is a tree, and a Scarf complex when G is saturated (see [13]); thus, the M_G form a natural family of ideals whose minimal free resolutions interpolate between these two extremes.

We now summarize the results of this paper in terms of the Betti numbers of M_G and I_G. A connected k-partition of G is a partition $\Pi = \sqcup_{j=1}^{k} V_j$ of $[n]$ such that the subgraphs induced by G on each set V_j is connected. The graph G_Π associated with the partition Π has vertices the elements of Π, and the (V_i, V_j)-entry of its adjacency matrix is $\sum_{u \in V_i, v \in V_j} a_{uv}$, where (a_{uv}) is the adjacency matrix for G. Let $\mathcal{P}_k$ be the set of connected k-partitions of G of size k.

Theorem 1.1 (Betti Numbers of M_G and I_G)**.** *Let G be an undirected connected multigraph. For a connected k-partition Π, let $\alpha(\Pi)$ denote the number of acyclic*

orientations on G_Π with a unique sink at the set containing vertex n. The Betti numbers of M_G and I_G are

$$\beta_k(R/M_G) = \beta_k(R/I_G) = \sum_{\Pi \in \mathcal{P}_{k+1}} \alpha(\Pi).$$

A few remarks on Theorem 1.1 are in place. The numbers $\beta_j(R/M_G)$ and $\beta_j(R/I_G)$ do not depend on the choice of the sink vertex n. To obtain a bijection between acyclic orientations of a graph with unique sink i and acyclic orientations of the same graph with unique sink j, simply reverse the orientations of all edges along paths from i to j. Natural bijections exist between the set of acyclic orientations with unique sink at a fixed vertex and the set of minimal recurrent configurations of a graph [4]. Hence, [16, Conjecture 7.9] follows from Theorem 1.1.

Let us close this section by providing an overview of the proof of Theorem 1.1.

Overview of proof of Theorem 1.1. We construct complexes $\mathcal{F}_1(G)$ and $\mathcal{F}_0(G)$ that are candidates for minimal free resolutions of I_G and M_G respectively (Sections 3 and 4). The ranks of the free modules at each homological degree match the description of the Betti numbers in Theorem 1.1. We are then left with proving the exactness of $\mathcal{F}_0(G)$ and $\mathcal{F}_1(G)$. We prove the exactness of $\mathcal{F}_0(G)$ in Section 5. We exploit the torus action on $\mathcal{F}_0(G)$ to reduce the exactness of $\mathcal{F}_0(G)$ to the exactness of certain complexes of vector spaces (Subsection 5.1) and prove the exactness of these complexes of vector spaces by decomposing them as a direct sum of certain complexes derived from Koszul complexes (Subsection 5.2). The proof of the exactness of $\mathcal{F}_1(G)$ uses the exactness of $\mathcal{F}_0(G)$ (Section 6). More precisely, we use Gröbner degeneration to derive a family of complexes $\mathcal{F}_t(G)$ parametrized by spec($\mathbb{K}[t]$) such that the fiber at (t) is $\mathcal{F}_0(G)$ and the fibers at $(t - t_0)$ for $t_0 \neq 0$ are all isomorphic to $\mathcal{F}_1(G)$. The integral weight function realizing the Gröbner degeneration is intimately connected to potential theory on graphs and one such choice is the function $b_q(D)$ studied in [3]. With this information at hand, we use well-known properties of flat families to deduce that $\mathcal{F}_1(G)$ is exact. □

Related results. Analogous results were obtained simultaneously and independently by Mohammadi and Shokrieh in [15], using different techniques. They give the Betti numbers for M_G and I_G and construct minimal free resolutions by using Schreyer's algorithm to explicitly compute the syzygies. As well, Horia Mania [12] has given an alternate proof that $\beta_1(R/I_G) = |\mathcal{P}_2|$ by computing the connected components of certain simplicial complexes associated with I_G.

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All examples were computed using the computer algebra systems Sage [18] and Macaulay 2 [9]. Those readers interested in computing their own examples can find a relevant script on the homepage of the third author, `http://johnwilmes.name/sand`.

2. Preliminaries

For the rest of this paper, G will denote an undirected connected multigraph with vertex set $[n]$. In other contexts, it is usual to describe multigraphs as having "multiple edges" joining adjacent vertices, but for our purposes it is more natural to think of a single edge with nonnegative integer weight. Since G is undirected, the adjacency matrix $A_G = (a_{ij})$ is symmetric.

We define minimal free resolutions of M_G and I_G in terms of acyclic partitions.

Definition 2.1. (**Acyclic Partition**) An **acyclic k-partition** $\mathcal{C}$ is a pair $(\Pi, \mathcal{A})$ where Π is a connected k-partition and $\mathcal{A}$ is an acyclic orientation on G_Π. We think of $\mathcal{C}$ as a directed graph on Π. Given a vertex i of G, the graph $\mathcal{C}$ is an **i-acyclic k-partition** if it has a unique sink at the element of Π containing i. If $\mathcal{C}$ is an acyclic k-partition, we denote by $\Pi(\mathcal{C})$ the corresponding connected k-partition of G.

Acyclic partitions are intimately related to the chip-firing game on a graph. This game consists of an initial configuration of an integer number D_j of chips at every vertex j of G. Such a configuration is called a *divisor* (cf. [2]), and is viewed as an element D of the free abelian group $\mathbb{Z}[V]$,

$$D = \sum_{j \in V} D_j \cdot j$$

where $V = [n]$ is the vertex set of G. The game is played by *firing* a vertex j, i. e., replacing the divisor D with $D - e_j \Lambda_G$, where e_j is the j^{th} standard basis vector. We say D is *linearly equivalent* to a divisor E if E can be reached from D by a sequence of such firings. Thus, viewed as elements of $\mathbb{Z}^n$, we have D and E linearly equivalent if and only if they are equivalent modulo the Laplacian lattice. For a more thorough introduction to the chip firing game, the reader is referred to [11].

Let $\mathcal{C}$ be an acyclic k-partition of G, and fix $u \in V(G)$. Let $U \in \Pi(\mathcal{C})$ be such that $u \in U$. Define $out_{\mathcal{C}}(u)$ as the number of edges in G between u and vertices appearing in sets which are out-neighbors of U in $\mathcal{C}$, i. e.,

$$out_{\mathcal{C}}(u) = \sum_{(U,W) \in \mathcal{C}} \sum_{w \in W} a_{uw}.$$

Given an acyclic k-partition $\mathcal{C}$ of G, we define a divisor $D(\mathcal{C})$ on G by

$$D(\mathcal{C}) = \sum_{v \in V(G)} out_{\mathcal{C}}(v) \cdot v.$$

A G-parking function (relative to n) is a divisor D on G with $D_n = -1$ such that if $A \subset V \setminus \{n\}$ and E is the divisor obtained from D by firing every vertex in A, then there is some vertex $i \in A$ such that $E_i < 0$. The divisors $D(\mathcal{C}) - 1 \cdot n$ for $\mathcal{C}$ an n-acyclic n-partition are exactly the maximal G-parking functions [4]. If D is a divisor on G with $D_n = -1$, then D is a G-parking function if and only if the monomial $\prod_{i<n} x_i^{D_i}$ is not in the G-parking function ideal M_G [17].

Definition 2.2. (**Chip Firing Equivalence**) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be acyclic k-partitions with $\Pi(\mathcal{C}_1) = \Pi(\mathcal{C}_2)$, and define the projection $p : \mathbb{Z}[V] \to \mathbb{Z}[\Pi(\mathcal{C}_1)]$ from divisors of G to divisors of $G_{\Pi(\mathcal{C}_1)}$ by $p(D)_U = \sum_{j \in U} D_j$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are chip firing equivalent, equivalent, denoted by $\mathcal{C}_1 \sim \mathcal{C}_2$, if the divisors $p(D(\mathcal{C}_1))$ and $p(D(\mathcal{C}_2))$ on $G_{\Pi(\mathcal{C}_1)}$ are linearly equivalent.

We write $[\mathcal{C}]$ for the chip firing equivalence class containing the acyclic k-partition $\mathcal{C}$. Each chip firing equivalence of acyclic n-partitions contains a unique n-acyclic k-partition, as a result of the well-known equivalence between G-parking functions and acyclic orientations (cf. [4]). The following lemma is an immediate generalization.

Lemma 2.3. *Every chip firing equivalence class of acyclic k-partitions contains a unique n-acyclic k-partition.*

Proof. For $\mathcal{C}$ an acyclic k-partition of G, write $\overline{\mathcal{C}}$ for the corresponding acyclic k-partition of $G_{\Pi(\mathcal{C})}$. Note that $p(D(\mathcal{C})) = D(\overline{\mathcal{C}})$. Thus, the lemma is immediate from the known result for acyclic n-partitions. □

Remark 1. If $\mathcal{C}$ is an acyclic k-partition with a source at $U \in \Pi(\mathcal{C})$, then by firing every vertex in U from $D(\mathcal{C})$ we obtain the divisor $D(\mathcal{C}')$, where $\mathcal{C}'$ is the acyclic k-partition with $\Pi(\mathcal{C}') = \Pi(\mathcal{C})$ given by reversing the orientation of every edge incident on U in $\mathcal{C}$, and preserving all other orientations (c.f. [8]). Similarly, we may turn a sink U into a source by firing all vertices not in U. Thus, if $\mathcal{C}$ and $\mathcal{C}'$ are acyclic k-partitions with $\Pi(\mathcal{C}) = \Pi(\mathcal{C}')$, then $\mathcal{C} \sim \mathcal{C}'$ if $\mathcal{C}'$ is obtained from $\mathcal{C}$ by iteratively replacing sources with sinks, or sinks with sources. The converse also holds, by the proof of Lemma 2.4 below. □

Lemma 2.4. *Let $\mathcal{C}$ and $\mathcal{C}'$ be acyclic k-partitions with $\Pi(\mathcal{C}) = \Pi(\mathcal{C}')$. Then if $\mathcal{C} \sim \mathcal{C}'$, the divisors $D(\mathcal{C})$ and $D(\mathcal{C}')$ are linearly equivalent.*

Proof. Suppose $p(D(\mathcal{C}))$ and $p(D(\mathcal{C}'))$ are linearly equivalent. By iteratively replacing sinks with sources in $\mathcal{C}$, as in Remark 1, we obtain an n-acyclic k-partition, and similarly for $\mathcal{C}'$. Thus, without loss of generality, we may assume $\mathcal{C}$ and $\mathcal{C}'$ are n-acyclic. But then $\mathcal{C} = \mathcal{C}'$ by Lemma 2.3. □

We remark that the converse of Lemma 2.4 also holds, though we shall not need it.

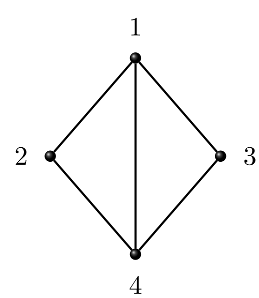

FIGURE 1. The "kite graph" on four vertices.

Example 1. Let G be the "kite graph" on four vertices depicted in Figure 1. Then G has a unique acyclic 1-partition, six chip-firing equivalence classes of acyclic 2-partitions, nine classes of acyclic 3-partitions, and four classes of acyclic 4-partitions. The 4-acyclic representatives of each of these is depicted in Figure 2.

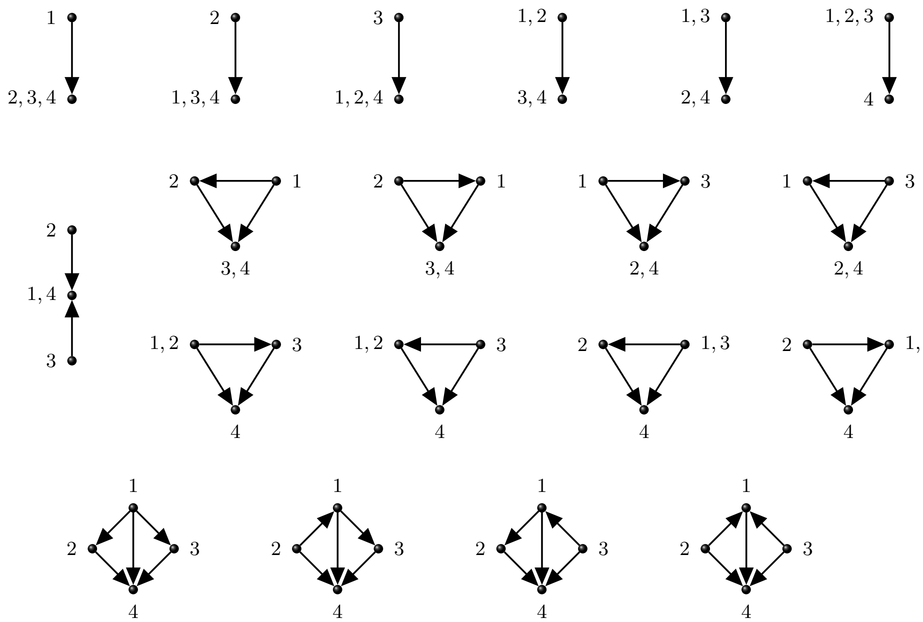

FIGURE 2. The 4-acyclic k-partitions of the "kite graph" for $k = 2, 3, 4$.

2.1. **Edge Contraction.** In order to define the differentials of our free resolutions of M_G and I_G, we will use the operation of edge contraction in acyclic partitions. For a directed edge $e = (A, B)$ of $\mathcal{C}$, we will denote by $e^- = A$ the tail of e, and by $e^+ = B$ the head of e.

Definition 2.5. (**Contractible Edge**) A (directed) edge e of an acyclic k-partition $\mathcal{C}$ is **contractible** if the directed graph $\mathcal{C}/e$ given by contracting e (i. e., merging the vertices e^+ and e^-) is acyclic. The edge e is a contractible edge of the chip firing equivalence class $\mathfrak{c}$ if there exists an acyclic k-partition $\mathcal{C} \in \mathfrak{c}$ such that e appears in $\mathcal{C}$ and furthermore e is contractible in $\mathcal{C}$.

Note that by the characterization of chip firing equivalence given in Remark 1, if e is a contractible edge of a chip firing equivalence class $\mathfrak{c}$ and e appears in $\mathcal{C} \in \mathfrak{c}$, then e is contractible in $\mathcal{C}$.

If $\mathcal{C}$ is an acyclic k-partition and e is a contractible edge of $\mathcal{C}$, then $\mathcal{C}/e$ is an acyclic $(k-1)$-partition—the sets e^- and e^+ in $\Pi(\mathcal{C})$ are replaced with the set $e^- \cup e^+$ in $\Pi(\mathcal{C}/e)$. If e and f are distinct edges of $\mathcal{C}$, and e is contractible, we write f/e for the edge corresponding to f in $\mathcal{C}/e$. When the graph we are referring to is clear from the context, we will sometimes abuse notation and write f instead of f/e. We define $[\mathcal{C}]/e$ to be $[\mathcal{C}/e]$. The class $[\mathcal{C}]/e$ is well-defined: suppose $\mathcal{C}'$ is an acyclic k-partition with $\Pi(\mathcal{C}') = \Pi(\mathcal{C})$ such that e is also contractible in $\mathcal{C}'$. Let

$$E = \sum_{u \in e^-} \sum_{v \in e^+} a_{uv} u.$$

By definition, $D(\mathcal{C}/e) = D(\mathcal{C}) - E$, and $D(\mathcal{C}'/e) = D(\mathcal{C}') - E$. Then $D(\mathcal{C}')$ is linearly equivalent to $D(\mathcal{C})$ if and only if $D(\mathcal{C}'/e)$ is linearly equivalent to $D(\mathcal{C}/e)$.

Definition 2.6. (**Monomial Associated with Edge Contraction**) Let $\mathfrak{c}$ be an equivalence class of acyclic k-partitions with contractible edge e. If e appears in $\mathcal{C} \in \mathfrak{c}$, we define the monomial $m_{\mathfrak{c}}(e) = \mathbf{x}^{D(\mathcal{C})-D(\mathcal{C}/e)}$.

The monomial $m_{\mathfrak{c}}(e)$ is well-defined, since if e also appears in $\mathcal{C}' \in \mathfrak{c}$, then $D(\mathcal{C}) - D(\mathcal{C}/e) = D(\mathcal{C}') - D(\mathcal{C}'/e)$.

Definition 2.7. (**Refinements of Acyclic Orientations**) Let ℓ, $k \in \mathbb{N}$. An equivalence class $\mathfrak{c}_1$ of acyclic k-partitions is called a **refinement** of an equivalence class $\mathfrak{c}_2$ of acyclic ℓ-partitions if $\mathfrak{c}_2$ is obtained from $\mathfrak{c}_1$ by some sequence of edge contractions.

Let $\mathfrak{c}_1$ be an equivalence class of acyclic k-partitions and $\mathfrak{c}_2$ an equivalence class of acyclic $(k-2)$-partitions. Suppose that there exists a contractible edge e of $\mathfrak{c}_1$ and a contractible edge f of $\mathfrak{c}_1/e$ such that $\mathfrak{c}_2 = (\mathfrak{c}_1/e)/f$. The following lemma states that we can lift f to a unique contractible edge of $\mathfrak{c}_1$, and then f and e are contractible in either order.

Lemma 2.8. *Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be equivalence classes of acyclic partitions such that $\mathfrak{c}_2 = (\mathfrak{c}_1/e)/f$ for some edges e and f. Then there is a unique edge g which is contractible relative to $\mathfrak{c}_1$ such that $g/e = f$. Furthermore, we have the following:*

1. *e/g is contractible relative to $\mathfrak{c}_1/g$.*
2. *$[(\mathfrak{c}_1/g)/e] = [(\mathfrak{c}_1/e)/g] = \mathfrak{c}_2$.*
3. *There exists some $\hat{\mathcal{C}} \in \mathfrak{c}_1$ in which both e and g appear (and hence are contractible).*

Proof. Let $\mathcal{C} \in \mathfrak{c}_1/e$ be such that f is contractible in $\mathcal{C}$, and lift $\mathcal{C}$ to an acyclic partition $\hat{\mathcal{C}} \in \mathfrak{c}_1$ by preserving the orientation of e, and orienting every other edge of $G_{\Pi(\mathfrak{c}_1)}$ as in $\mathcal{C}$. If there is only one edge $g \in \hat{\mathcal{C}}$ such that $g/e = f$, we are done. If there are two such edges (i. e., when either f^- or f^+ is equal to $e^+ \cup e^-$), then only one of them is contractible: if $f^- = e^+ \cup e^-$ then (e^+, f^+) is not contractible in $\hat{\mathcal{C}}$ but (e^-, f^+) is, and similarly if $f^+ = e^+ \cup e^-$. In all cases, there is exactly one contractible edge $g \in \hat{\mathcal{C}}$ such that $g/e = f$, and the result follows. □

We now define functions $\operatorname{sign}_{\mathfrak{c}}$ for every equivalence class $\mathfrak{c}$ of acyclic k-partitions, taking the contractible edges of $\mathfrak{c}$ to $\{\pm 1\}$. We choose these maps so that if e and f are distinct contractible edges of $\mathfrak{c}$, then

$$\operatorname{sign}_{\mathfrak{c}}(e) \operatorname{sign}_{\mathfrak{c}/e}(f) = -\operatorname{sign}_{\mathfrak{c}}(f) \operatorname{sign}_{\mathfrak{c}/f}(e). \tag{1}$$

Furthermore, we insist that if both $e = (A, B)$ and $\hat{e} = (B, A)$ are contractible edges of $\mathfrak{c}$ for some sets $A, B \in \Pi(\mathfrak{c})$, then

$$\operatorname{sign}_{\mathfrak{c}}(e) = -\operatorname{sign}_{\mathfrak{c}}(\hat{e}). \tag{2}$$

Proposition 2.9. *A function $\operatorname{sign}_{\mathfrak{c}}$ satisfying (1) and (2) exists.*

Proof. For every equivalence class $\mathfrak{c}$ of acyclic k-partitions, fix a total ordering $\tau_{\mathfrak{c}}$ of $\Pi(\mathfrak{c})$. If τ is a total ordering of a set C, denote by $\operatorname{pos}_\tau(c)$ the position of c in this total ordering for any $c \in C$.

Let $\mathfrak{c}$ be an equivalence class of acyclic k-partitions, and let e be a contractible edge of $\mathfrak{c}$. Let ρ be a total ordering of $\Pi(\mathfrak{c})$ such that $\operatorname{pos}_\rho(e^-) = 0$ and $\operatorname{pos}_\rho(e^+) = 1$. Let ρ/e be the total ordering of $\Pi(\mathcal{C}/e)$ with $\operatorname{pos}_{\rho/e}(e^- \cup e^+) = 0$, and $A <_{\rho/e} B$ if $A <_\rho B$ for all other sets. Let $\operatorname{sign}(\rho)$ and $\operatorname{sign}(\rho/e)$ denote the signs of the

permutation taking ρ to $\tau_{\mathfrak{c}}$, and ρ/e to $\tau_{\mathfrak{c}/e}$, respectively. We define $\operatorname{sign}_{\mathfrak{c}}(e) = \operatorname{sign}(\sigma)\operatorname{sign}(\sigma/e)$.

The function $\operatorname{sign}_{\mathfrak{c}}$ does not depend on the choice of ρ; indeed, if we take another total ordering ρ' of $\Pi(\mathcal{C})$ for which $\operatorname{pos}_{\rho'}(e^-) = 0$ and $\operatorname{pos}_{\rho'}(e^+) = 0$, then the sign of the permutation taking ρ' to ρ is the same as the sign of the permutation taking ρ'/e to ρ/e.

Clearly $\operatorname{sign}_{\mathfrak{c}}$ satisfies (2). To verify (1) for contractible edges e, f of $\mathfrak{c}$, there are four cases to consider: (i) $e^- \cup e^+$ and $f^- \cup f^+$ are disjoint; (ii) $e^- = f^-$; (iii) $e^+ = f^+$; and (iv) $e^- = f^+$. The argument for all four cases is similar. For example, in case (i) we consider a total ordering ρ of $\Pi(\mathfrak{c})$ for which the first four elements are e^-, e^+, f^-, f^+, in that order, and compute signs by contracting e and f in either order. □

3. Minimal free resolution of I_G

We now define the complex $\mathcal{F}_1(G)$ that, as we will show, is a minimal free resolution for I_G. For an equivalence class of acyclic $(k+1)$-partitions $\mathfrak{c}$, let $D(\mathfrak{c}) \in \mathbb{Z}^n/\Lambda_G$ denote the linear equivalence class of divisors corresponding to the elements of $\mathfrak{c}$. For k from 0 to $n-1$, define the k^{th} homological degree of $\mathcal{F}_1(G)$ to be the free module

$$F_{1,k} = \bigoplus_{\mathfrak{c}} R(-D(\mathfrak{c})),$$

where the direct sum is taken over all chip-firing equivalence classes of acyclic $(k+1)$-partitions $\mathfrak{c}$, which are identified with the standard basis elements of $F_{1,k}$, and $R(-D(\mathfrak{c}))$ denotes the twist of R by $-D(\mathfrak{c})$. Now we define differentials $\delta_{1,k} : F_{1,k+1} \to F_{1,k}$ of $\mathcal{F}_1(G)$ by the equations

$$\delta_{1,k}(\mathfrak{c}) = \sum_{e} \operatorname{sign}_{\mathfrak{c}}(e) m_{\mathfrak{c}}(e) \cdot (\mathfrak{c}/e) \tag{3}$$

where the sum is taken over contractible edges of $\mathfrak{c}$. Then $\mathcal{F}_1(G)$ is the sequence

$$\mathcal{F}_1(G) : F_{1,n-1} \xrightarrow{\delta_{1,n-1}} \cdots \xrightarrow{\delta_{1,2}} F_{1,1} \xrightarrow{\delta_{1,1}} F_{1,0}.$$

Example 2. For the "kite graph" G depicted in Figure 1, the complex $\mathcal{F}_1(G)$ reads as follows:

$$\mathcal{F}_1(G) : R^4 \xrightarrow{\delta_{1,3}} R^9 \xrightarrow{\delta_{1,2}} R^6 \xrightarrow{\delta_{1,1}} R^1.$$

The matrices of differentials are

$$\delta_{1,1} = \begin{pmatrix} x_1^3 - x_2x_3x_4 & x_2^2 - x_1x_4 & x_3^2 - x_1x_4 & x_1^2x_2 - x_3x_4^2 & x_1^2x_3 - x_2x_4^2 & x_1x_2x_3 - x_4^3 \end{pmatrix}$$

$$\delta_{1,2} = \begin{pmatrix} 0 & -x_2 & -x_4 & -x_3 & -x_4 & 0 & 0 & 0 & 0 \\ -x_3^2 + x_1x_4 & -x_3x_4 & -x_1^2 & 0 & 0 & 0 & 0 & -x_4^2 & -x_1x_3 \\ x_2^2 - x_1x_4 & 0 & 0 & -x_2x_4 & -x_1^2 & -x_4^2 & -x_1x_2 & 0 & 0 \\ 0 & x_1 & x_2 & 0 & 0 & -x_3 & -x_4 & 0 & 0 \\ 0 & 0 & 0 & x_1 & x_3 & 0 & 0 & -x_2 & -x_4 \\ 0 & 0 & 0 & 0 & 0 & x_1 & x_3 & x_1 & x_2 \end{pmatrix}$$

$$\delta_{1,3} = \begin{pmatrix} -x_4 & 0 & 0 & x_1 \\ x_3 & 0 & -x_4 & 0 \\ 0 & x_3 & 0 & x_4 \\ -x_2 & -x_4 & 0 & 0 \\ 0 & 0 & x_2 & -x_4 \\ x_1 & x_2 & 0 & 0 \\ 0 & 0 & -x_1 & x_2 \\ -x_1 & 0 & x_3 & 0 \\ 0 & -x_1 & 0 & -x_3 \end{pmatrix}.$$

The basis elements of the free modules in $\mathcal{F}_1(G)$ correspond to the six chip firing equivalence classes of acyclic 2-partitions, nine chip firing equivalence classes of acyclic 3-partitions and four chip firing equivalence classes of acyclic 4-partitions, in the order (from left to right) depicted in Figure 2. $\square$

The first main result of this paper is the following:

Theorem 3.1. *$\mathcal{F}_1(G)$ is a minimal free resolution of I_G.*

We remark that $\mathcal{F}_1(G)$ is naturally graded by $\mathbb{Z}^n/\Lambda_G$. In Lemma 3.2 below, we will show that $\mathcal{F}_1(G)$ is complex and that the cokernel of $\delta_{1,1}$ is equal to R/I_G. We will complete the proof of Theorem 3.1 in Section 6 where we establish the exactness of $\mathcal{F}_1(G)$.

Lemma 3.2. *$\mathcal{F}_1(G)$ is a complex of free R-modules, and the cokernel of $\delta_{1,1}$ is equal to R/I_G.*

Proof. First, we show that the cokernel of $\delta_{1,1}$ is equal to R/I_G. If $\mathfrak{c}$ is an equivalence class of acyclic 2-partitions, then the two elements $\mathcal{C}_1, \mathcal{C}_2$ of $\mathfrak{c}$ are the two possible orientations of an edge $\{A, \overline{A}\}$, where both A and $\overline{A}$ induce connected subgraphs of G. Then

$$\delta_{1,1}(\mathfrak{c}) = \pm(\mathbf{x}^{D(\mathcal{C}_1)} - \mathbf{x}^{D(\mathcal{C}_2)}) = \pm(\mathbf{x}^{S\to\overline{S}} - \mathbf{x}^{\overline{S}\to S})$$

which lies in I_G since $D(\mathcal{C}_1)$ and $D(\mathcal{C}_2)$ are linearly equivalent. Furthermore, by [13, Theorem 25], (following [5, Theorem 14]), the binomials

$$\mathbf{x}^{S\to\overline{S}} - \mathbf{x}^{\overline{S}\to S},$$

where both S and $\overline{S}$ are connected, form a Gröbner basis for I_G, and in particular they generate I_G.

Now we show that the $\delta_{1,k}$ are differentials. Fix an equivalence class $\mathfrak{c}$ of acyclic k-partitions of G, with $k \geq 2$. We wish to show that for any equivalence class $\mathfrak{c}'$ of acyclic $(k-2)$-partitions of G, the $\mathfrak{c}'$ component of $\delta_{1,k-1}(\delta_{1,k}(\mathfrak{c}))$ is 0. A nonzero term appearing in the $\mathfrak{c}'$ component of $\delta_{1,k-1}(\delta_{1,k}(\mathfrak{c}))$ results from a sequence of two edge contractions, say a contractible edge e of $\mathfrak{c}$ and a contractible edge f of

$\mathfrak{c}/e$. By Lemma 2.8, there exists a unique edge g of $\mathfrak{c}$ such that g is contractible and $g/e = f$. Furthermore, $\mathfrak{c}' = (\mathfrak{c}/g)/e$. Thus, it suffices to show that

$$\operatorname{sign}_{\mathfrak{c}}(e)\operatorname{sign}_{\mathfrak{c}/e}(g)m_{\mathfrak{c}}(e)m_{\mathfrak{c}/e}(g) + \operatorname{sign}_{\mathfrak{c}}(g)\operatorname{sign}_{\mathfrak{c}/g}(e)m_{\mathfrak{c}}(g)m_{\mathfrak{c}/g}(e) = 0.$$

By Property (1) of $\operatorname{sign}_{\mathfrak{c}}$, it suffices to show that

$$m_{\mathcal{C}}(e)m_{\mathcal{C}/e}(g) = m_{\mathcal{C}}(g)m_{\mathcal{C}/g}(e).$$

Let $\mathcal{C} \in \mathfrak{c}$ be such that both e and g appear in $\mathcal{C}$. Note that such an acyclic k-partition is guaranteed by Lemma 2.8. We have

$$m_{\mathcal{C}}(e)m_{\mathcal{C}/e}(g) = \mathbf{x}^{D(\mathcal{C})-D(\mathcal{C}/e)}\mathbf{x}^{D(\mathcal{C}/e)-D((\mathcal{C}/e)/g)} = \mathbf{x}^{D(\mathcal{C})-D((\mathcal{C}/e)/g)}$$

and similarly $m_{\mathcal{C}}(g)m_{\mathcal{C}/g}(e) = \mathbf{x}^{D(\mathcal{C})-D((\mathcal{C}/e)/g)}$. □

Remark 2. When viewed as a matrix with entries over the polynomial ring R, the nonzero entries of $\delta_{1,k}$ are either monomials or binomials. Let $\mathfrak{c}$ be an equivalence class of acyclic $(k+1)$-partitions, and $\mathfrak{c}'$ an equivalence class of acyclic k-partitions. Suppose some contractible edge e of $\mathfrak{c}$ satisfies $\mathfrak{c}/e = \mathfrak{c}'$. Then $\Pi(\mathfrak{c}')$ is obtained from $\Pi(\mathfrak{c})$ by replacing e^- and e^+ with $e^- \cup e^+$. Hence, there is at most one other directed edge $\hat{e}$ such that $\mathfrak{c}/\hat{e} = \mathfrak{c}'$, namely $\hat{e} = (e^+, e^-)$.

In fact, $\mathfrak{c}/e = \mathfrak{c}/\hat{e}$ if and only if the edge $\{e^-, e^+\}$ of $\mathfrak{c}$ is a bridge of $G_{\Pi(\mathfrak{c})}$. If the edge is a bridge between sets A and B, and the edge is oriented from A to B in some acyclic $(k+1)$-partition $\mathcal{C} \in \mathfrak{c}$, then we can fire A to obtain a chip firing equivalent acyclic $(k+1)$-partition which differs from $\mathcal{C}$ only on the orientation of this edge. It follows that $\mathfrak{c}/e = \mathfrak{c}/\hat{e}$. Similarly, if $\mathfrak{c}/e = \mathfrak{c}/\hat{e} = \mathfrak{c}'$, then fix $\mathcal{C} \in \mathfrak{c}'$. We can lift $\mathcal{C}$ to $\mathfrak{c}$ by introducing the edge $\{e^-, e^+\}$ in either orientation, so the two resulting acyclic $(k+1)$-partitions are chip firing equivalent. By Remark 1, we may obtain one acyclic $(k+1)$-partition from the other by iteratively turning sinks into sources by reversing edges. Since no edge other than e is reversed at the end of this process, it follows that e does not participate in any cycles. Thus, binomial entries in $\delta_{1,k}$ correspond to bridges of partition graphs, i. e., cuts of G.

4. Minimal free resolution of M_G

We will now define the complex $\mathcal{F}_0(G)$, the minimal free resolution for M_G. As we will show later in this section, in the case when G is a tree, $\mathcal{F}_0(G)$ is a Koszul complex.

For k from 0 to $n-1$, define the k^{th} homological degree of $\mathcal{F}_0(G)$ to be the free module

$$F_{0,k} = \bigoplus_{\mathcal{C}} R(-D(\mathcal{C}))$$

where the direct sum is taken over all n-acyclic $(k+1)$-partitions $\mathcal{C}$ of G, which are identified with the standard basis elements of $F_{0,k}$. Now we define differentials $\delta_{0,k} : F_{0,k} \to F_{0,k-1}$ of $\mathcal{F}_0(G)$ by the equations

$$\delta_{0,k}(\mathcal{C}) = \sum_{e} \operatorname{sign}_{\mathcal{C}}(e)m_{\mathcal{C}}(e) \cdot (\mathcal{C}/e) \tag{4}$$

where the sum is taken over contractible edges of $\mathcal{C}$. Then $\mathcal{F}_0(G)$ is the sequence

$$\mathcal{F}_0(G) : F_{0,n-1} \xrightarrow{\delta_{0,n-1}} \cdots \xrightarrow{\delta_{0,2}} F_{0,1} \xrightarrow{\delta_{0,1}} F_{0,0}.$$

We emphasize that the essential difference between $\mathcal{F}_1(G)$ and $\mathcal{F}_0(G)$ is that in the differentials of the latter, the sum is taken only over contractible edges of an

n-acyclic $(k+1)$-partition, rather than over all contractible edges of a chip-firing equivalence class of acyclic partitions. Since edges only appear in one orientation in the n-acyclic partition of chip-firing equivalence class, the condition in Equation (2) is no longer relevant. For the maps $\mathrm{sign}_{\mathcal{C}}$, we require only Property (1) to hold. Finally, we note that whereas $\mathcal{F}_1(G)$ was graded by $\mathbb{Z}^n/\Lambda_G$, the complex $\mathcal{F}_0(G)$ is graded by $\mathbb{N}^{n-1}$.

Example 3. For the "kite graph" G depicted in Figure 1, the complex $\mathcal{F}_0(G)$ reads as follows:

$$\mathcal{F}_0(G) : R^4 \xrightarrow{\delta_{0,3}} R^9 \xrightarrow{\delta_{0,2}} R^6 \xrightarrow{\delta_{0,1}} R^1.$$

The matrices of differentials are

$$\delta_{0,1} = \begin{pmatrix} x_1^3 & x_2^2 & x_3^2 & x_1^2x_2 & x_1^2x_3 & x_1x_2x_3 \end{pmatrix}$$

$$\delta_{0,2} = \begin{pmatrix} 0 & -x_2 & 0 & -x_3 & 0 & 0 & 0 & 0 & 0 \\ -x_3^2 & 0 & -x_1^2 & 0 & 0 & 0 & 0 & 0 & -x_1x_3 \\ x_2^2 & 0 & 0 & 0 & -x_1^2 & 0 & -x_1x_2 & 0 & 0 \\ 0 & x_1 & x_2 & 0 & 0 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & x_3 & 0 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & x_3 & x_1 & x_2 \end{pmatrix}$$

$$\delta_{0,3} = \begin{pmatrix} 0 & 0 & 0 & x_1 \\ x_3 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 \\ -x_2 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ x_1 & x_2 & 0 & 0 \\ 0 & 0 & -x_1 & x_2 \\ -x_1 & 0 & x_3 & 0 \\ 0 & -x_1 & 0 & -x_3 \end{pmatrix}.$$

The basis elements of the free modules in $\mathcal{F}_1(G)$ correspond to the six 4-acyclic 2-partitions, nine 4-acyclic 3-partitions, and four 4-acyclic 4-partitions, in the order (from left to right) depicted in Figure 2. $\square$

The second main result of this paper is that $\mathcal{F}_0(G)$ is a minimal free resolution of M_G.

Theorem 4.1. *$\mathcal{F}_0(G)$ is a minimal free resolution of M_G.*

The proof that $\mathcal{F}_0(G)$ is a complex of free R-modules and the cokernel of $\delta_{0,1}$ is equal to R/M_G proceeds along the lines of the analogous statement for $\mathcal{F}_1(G)$. We will complete the proof of Theorem 4.1 in Section 5 where we establish the exactness of $\mathcal{F}_0(G)$. In the rest of this section, we will show that for any tree T, the complex $\mathcal{F}_0(T)$ is isomorphic to the Koszul complex. Since M_T is the irrelevant ideal $\langle x_1, \dots, x_{n-1}\rangle$ it follows that $\mathcal{F}_0(T)$ is a minimal free resolution of M_T.

Lemma 4.2. *Let $R = \mathbb{K}[x_1, \dots, x_n]$ and let*

$$\mathcal{K} : K_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} K_1 \xrightarrow{\delta_1} K_0$$

be the Koszul complex for $(x_1, \dots, x_n)$. Suppose $\delta_i' : K_{i+1} \to K_i$ are differentials which agree with δ_i as monomial matrices up to the signs of their entries. Then for some collection B of basis elements of each of the K_i, we can obtain δ from δ' by

composing with the map sending each element of B to its negative. In particular, the complex given by the δ'_i is exact.

Proof. By induction on i. Suppose $\phi : K_{i-1} \to K_{i-1}$ is given by sending some collection of basis elements to their negative, and $\delta'_{i-1} \circ \phi = \delta_{i-1}$. Write $\phi \circ \delta'_i$ and δ_i as matrices over R. We wish to show that these matrices are equal up to multiplying some collection of columns by -1. Let u be the column of $\phi \circ \delta'_i$ corresponding to basis element e, and v the e-column of δ_i. Since $\delta'_{i-1} \circ \phi u = \delta_{i-1} u = \vec{0}$, and since $\mathcal{K}$ is exact, it follows that u is an R-linear combination of the columns of δ_i. But the nonzero entries of u are monomials of degree one, which agree with v up to their sign. Furthermore, by the definition of the Koszul complex, every variable x_i appears at most once in each row of δ_i. It follows that $u = \pm v$. $\square$

Lemma 4.3. *Let G be a tree. Then $M_G = \langle x_1, \dots, x_{n-1} \rangle$, and $\mathcal{F}_0(G)$ is a minimal free resolution of R/M_G. (In fact, $\mathcal{F}_0(G)$ is a Koszul complex.)*

Proof. If G is a tree, and $\mathcal{C}$ is an n-acyclic k-partition, then let $\Delta_{\mathcal{C}}$ denote the subset of $\{1, \dots, n-1\}$ given by those vertices of G with an out-edge appearing in $\mathcal{C}$. Then in characteristic two, $\mathcal{F}_0(G)$ is isomorphic to the Koszul complex via the map sending $e_{\mathcal{C}}$ to the basis element of the Koszul complex corresponding to $\Delta_{\mathcal{C}}$. Use Lemma 4.2 and the fact that $R/\langle x_1, \dots, x_{n-1} \rangle$ is minimally resolved by the Koszul complex to conclude that $\mathcal{F}_0(G)$ minimally resolves R/M_G. $\square$

5. Exactness of $\mathcal{F}_0(G)$

In this section we will establish the exactness of $\mathcal{F}_0(G)$. In Subsection 5.1, we reduce the exactness of $\mathcal{F}_0$ to the exactness of certain complexes of vector spaces and in Subsection 5.2, we show that these complexes of vector spaces are exact.

5.1. **Reduction to a Complex of Vector Spaces.** For a prime ideal P of R, denote by $\kappa(P)$ the residue field R_P/P at P.

Lemma 5.1. *Let P_j be the ideal $\langle x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1} \rangle$ of $\mathbb{K}[x_1, \dots, x_{n-1}]$. For the complex $\mathcal{F}_0$, the following statements are equivalent:*

1. *$\mathcal{F}_0$ is exact.*
2. *The complexes $(\mathcal{F}_0)_P$ and $(\mathcal{F}_0)_P \otimes \kappa(P)$ are split exact for all prime ideals P of $\mathbb{K}[x_1, \dots, x_{n-1}]$ except the irrelevant ideal $\langle x_1, \dots, x_{n-1} \rangle$.*
3. *For each j from 1 to $n-1$, the complex $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ is split exact as a complex of vector spaces.*

Proof. $(1 \Rightarrow 2)$ Since exactness is a local property and M_G is an Artinian monomial ideal, the complex $(\mathcal{F}_0)_P$ is, in fact, split exact for all prime ideals P. Hence, for all $i \geq 1$ the modules $\mathrm{Tor}^i((\mathcal{F}_0)_P, \kappa(P))$ are zero. This shows that $(\mathcal{F}_0)_P \otimes \kappa(P)$ is also split exact.
$(2 \Rightarrow 3)$ Note that P_j is a prime ideal for all integers j from 1 to n.
$(3 \Rightarrow 1)$ We first show that if $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ is split exact as a complex of vector spaces over the residue field $\kappa(P_j)$ of the local ring at P_j, then $(\mathcal{F}_0)_{P_j}$ is split exact by the following argument.

Take an element b in $(F_{0,0})_{P_j}$ and consider its projection b_p in $(F_{0,0})_{P_j} \otimes \kappa(P_j)$. Since $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ is split exact, b_p is contained image of the first differential of $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$. Using the map between the complexes $(\mathcal{F}_0)_{P_j}$ and $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ and the fact that the natural projection from $(F_{0,0})_{P_j}$ to $(F_{0,0})_{P_j} \otimes \kappa(P_j)$ is

surjective, we know that there is an element c in $F_{0,1}$ such that $\delta_{0,1}(c)$ is equal to b modulo $\mathfrak{m} \cdot (F_{0,0})_{P_j}$ where $\mathfrak{m}$ is the unique maximal ideal of the local ring R_{P_j}. Hence, $\operatorname{im} \delta_{0,1} + \mathfrak{m} \cdot (F_{0,0})_{P_j} = (F_{0,0})_{P_j}$. By Nakayama's lemma, $\operatorname{im} \delta_{0,0} = (F_{0,0})_{P_j}$ and hence, $\delta_{0,0}$ is surjective with $\ker \delta_{0,1} \oplus (F_{0,0})_{P_j} = (F_{0,1})_{P_j}$. This shows that $\ker \delta_{0,1}$ is a projective module over a local ring and hence $\ker \delta_{0,1}$ is also free. Thus, we can write $(\mathcal{F}_0)_{P_j}$ as a direct sum of a trivial complex $(F_{0,0})_{P_j} \xrightarrow{id} (F_{0,0})_{P_j}$ and another complex $\mathcal{F}'$. We iterate the argument on $\mathcal{F}'$ to deduce that $(\mathcal{F}_0)_{P_j}$ is a trivial complex and is therefore split exact.

We now suppose that $(\mathcal{F}_0)_{P_j}$ is split exact and show that the sheafified complex $\tilde{\mathcal{F}}_0$ (over $\mathrm{Proj}R$) is exact. Using the open property of exactness, we deduce that the set L of points $\mathbb{P}^{n-2}$ whose stalks are not exact is a Zariski closed set. Since L is Zariski closed and since the module M_G is invariant under the torus action, L is also invariant under the action of the algebraic torus $(\overline{\mathbb{K}}^*)^{n-1}$. Every closed set invariant under the torus action is the zero set of a monomial ideal and hence, if nonempty, it contains one of the coordinate points e_j, and indeed the vanishing ideal of e_j is P_j. Hence, we deduce that the stalk of $\tilde{\mathcal{F}}_0$ at every closed point of $\mathbb{P}^{n-2}$ is exact and hence $\tilde{\mathcal{F}}_0$ is exact. Thus, the support of the homology modules of $\mathcal{F}_0$ is either empty or the irrelevant maximal ideal $\langle x_1, \ldots, x_n \rangle$. Hence, the Krull dimension of the nonzero homology module of $\mathcal{F}_0$ is zero (since the Krull dimension of a module is by definition the dimension of its support). Since the depth is at most the Krull dimension, the depth of the homology modules of $\mathcal{F}_0$ are all zero. We now note the hypothesis for the acyclicity lemma of Peskin and Szpiro [7, Lemma 20.11] are all satisfied: we have a graded ring R, the complex $\mathcal{F}_0$ has length at most the depth of R and the depth of $F_{0,k}$ is at least k (actually in this case $F_{0,k}$ has depth n). Hence, we deduce that the homology modules of $\mathcal{F}_0$ are all zero and $\mathcal{F}_0$ is exact. $\square$

Note that the matrix of the differentials of $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ can be obtained by substituting one for x_j and zero for all other indeterminates in the corresponding matrix of differentials of $\mathcal{F}_0$.

5.2. **Exactness of the Complex of Vector Spaces.** In this section, we show that $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ is exact for any j from 1 to $n-1$. To that end, we will decompose $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ into a direct sum of complexes of vector spaces arising from localized Koszul complexes. In fact, the Koszul complexes will be the complexes $\mathcal{F}_0(H)$ for certain star graphs H which we will now define.

Consider the differential $\delta^*_{k,j} = (\delta_{0,k})_{P_j} \otimes \kappa(P_j)$ induced from the differential $\delta_{0,k}$ of $\mathcal{F}_0$. We represent $\delta^*_{k,j}$ as a zero-one matrix with respect to the standard basis elements with every one in the matrix corresponding to merging vertices V_r and V_q of an n-acyclic $(k+1)$-partition $\mathcal{C}$ such that: (i) $j \in V_r$, and (ii) any edge of G with one vertex in V_r and the other vertex in V_q is incident on j. Say an edge (V_r, V_q) of $\mathcal{C}$ is a *j-edge* if it is contractible, and satisfies (i) and (ii), and consider the subgraph $\mathcal{S}$ of $\mathcal{C}$ composed of the j-edges, and all vertices incident with these edges. Thus, $\mathcal{S}$ is a star, and every nonzero entry in the $e_{\mathcal{C}}$-column of $\delta^*_{k,j}$ corresponds to an edge of $\mathcal{S}$. We say $\mathcal{S}$ is the *j-star* associated with $\mathcal{C}$, and denote it by $\mathcal{S}(\mathcal{C})$. It is important to clarify that $\mathcal{C}$ is part of the data of $\mathcal{S}(\mathcal{C})$; even if the j-edges of two distinct n-acyclic $(k+1)$-partitions $\mathcal{C}$ and $\mathcal{C}'$ are identical, we do not identify $\mathcal{S}(\mathcal{C})$ with $\mathcal{S}(\mathcal{C}')$.

If $e_{\mathcal{C}}$ appears in a term in the image of $\delta^*_{k+1,j}$, it follows that $\mathcal{C}$ is obtained by contracting a j-edge of some n-acyclic $(k+1)$-partition $\mathcal{C}'$. In fact, there is a unique maximal such refinement of $\mathcal{C}$.

Proposition 5.2. *Let $\mathcal{C}$ be an n-acyclic k-partition, and let ℓ be maximal such that there exists an n-acyclic ℓ-partition $\mathcal{C}'$ with $\mathcal{C}$ obtained from $\mathcal{C}'$ be a sequence of contractions of j-edges. Then $\mathcal{C}'$ is the unique such n-acyclic ℓ-partition.*

Now suppose $\mathcal{C}'$ is the unique maximal refinement of $\mathcal{C}$ by j-edges, as in Proposition 5.2. If $\mathcal{C}' = \mathcal{C}$, we say the star $S(\mathcal{C})$ is *maximal*. In any case, if $\mathcal{C}$ is obtained from $\mathcal{C}'$ by contracting a collection E of edges, then $\mathcal{S}(\mathcal{C})$ is obtained from $\mathcal{S}(\mathcal{C}')$ by contracting the same collection of edges.

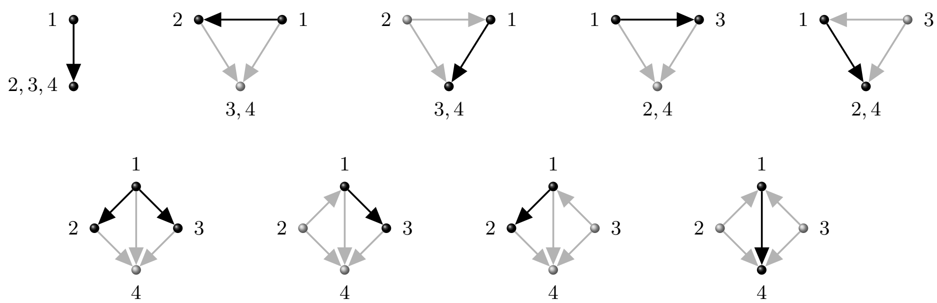

FIGURE 3. The maximal 1-stars associated with the "kite graph." The edges and vertices participating in each star are in black, and the rest are grayed out. The 4-acyclic partitions whose 1-stars are not maximal are not shown.

Suppose $\mathcal{S}$ is a maximal j-star associated to some n-acyclic $(k+1)$-partition $\mathcal{C}'$. We associate to $\mathcal{S}$ a complex $\mathcal{H}(\mathcal{S})$ of vectors as follows. Let m be the number of vertices of $\mathcal{S}$, and relabel the vertices of $\mathcal{S}$ with $[m]$ so that the vertex containing j receives the label m. Consider the G-parking function ideal $M_{\mathcal{S}}$ over the ring $\mathbb{K}[x_1, \ldots, x_m]$. We define $\mathcal{H}(\mathcal{S})$ to be the complex $\mathcal{F}_0(\mathcal{S})_{\langle x_m \rangle} \otimes \kappa(\langle x_m \rangle)$, with zeroes appended on either end so that the $(m-1)^{\text{th}}$ homological degree of $\mathcal{F}_0(\mathcal{S})$ corresponds to the k^{th} homological degree of $\mathcal{H}(\mathcal{S})$ (see Example 4). As a matrix, the map of vectors spaces $\gamma_\ell = (\delta_{0,\ell})_{\langle x_m \rangle} \otimes \kappa(\langle x_m \rangle)$ is obtained from the differential map for $\mathcal{F}_0(\mathcal{S})$ by replacing all indeterminates with 1. (The indeterminate x_m never appears in these differentials.)

To fully specify $\mathcal{F}_0(\mathcal{S})$, and hence $\mathcal{H}(\mathcal{S})$, over fields of arbitrary characteristic, we need to define the sign functions $\mathrm{sign}_{\mathcal{A}}$ for every m-acyclic ℓ-partition $\mathcal{A}$ of $\mathcal{S}$. The basis elements for the free module in homological degree k of $\mathcal{H}(\mathcal{S})$ correspond to j-stars $\mathcal{S}(\mathcal{C})$ associated to some n-acyclic k-partition $\mathcal{C}$ refined by $\mathcal{C}'$. The contractible edges of $\mathcal{A}$ correspond to j-edges of $\mathcal{C}$. Thus, we define $\mathrm{sign}_{\mathcal{A}}(e) = \mathrm{sign}_{\mathcal{C}}(e)$. Clearly $\mathrm{sign}_{\mathcal{A}}(e)$ satisfies (1), since $\mathrm{sign}_{\mathcal{C}}(e)$ does. Now recall that $\mathcal{F}_0(\mathcal{S})$ is exact by Lemma 4.3. Thus, by Lemma 5.1, $\mathcal{H}(S)$ is exact as well.

Let $\mathcal{M} = \oplus_{\mathcal{S}} \mathcal{H}(\mathcal{S})$, where the direct sum is taken over all maximal j-stars of G. Define maps ϕ_k from the vector space of $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ at homological degree k to the corresponding vector space of $\mathcal{M}$ by sending the n-acyclic $(k+1)$-partition $\mathcal{C}$ to its j-star $\mathcal{S}(\mathcal{C})$. These maps are isomorphisms of vector spaces by construction.

In addition, the maps ϕ_k commute with the differentials of $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ and $\mathcal{M}$, since edge contractions in the stars correspond exactly to edge contractions in G giving nonzero entries in the differentials of $(\mathcal{F}_0) \otimes \kappa(P_j)$, and the signs associated with these contractions are equal. Hence, we have:

Lemma 5.3. *For any integer j from $1, \ldots, n-1$, the complex $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ is isomorphic as a complex of $\mathbb{K}$-vectors spaces to $\mathcal{M}$. In particular, since $\mathcal{H}(\mathcal{S})$ is split exact for every maximal star $\mathcal{S}$, the complex $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ is split exact as well.*

Using Lemma 5.3 and Lemma 5.1, we complete the proof of Theorem 4.1.

Proof of Theorem 4.1. As we observed in the discussion following the statement of Theorem 4.1, $\mathcal{F}_0(G)$ is a complex and the cokernel of $\delta_{0,1}$ is R/M_G. By Lemma 5.3, we know that $(\mathcal{F}_0)_{P_j} \otimes \kappa(P_j)$ is split exact for every integer j with $1 \leq j \leq n-1$. It then follows by Lemma 5.1 that $\mathcal{F}_0(G)$ is exact, so that $\mathcal{F}_0(G)$ is a free resolution for M_G. Furthermore, since the image of $\delta_{0,k}$ is in $\langle x_1, \ldots, x_{n-1} \rangle F_{0,k-1}$ for every integer k, it follows that $\mathcal{F}_0(G)$ is minimal. $\square$

As a corollary to maximal star decomposition obtained in Theorem 5.3, we obtain another formula for Betti numbers of M_G:

Corollary 5.4. *Fix an integer j from $1, \ldots, n-1$. For integers $0 \leq s, t \leq n$, let $q_{s,t}$ denote the number of maximal j-stars of G having t vertices and corresponding to n-acyclic $(s+1)$-partitions. Then for every integer $0 \leq k \leq n$, we have $\beta_k(R/I_G) = \beta_k(R/M_G) = \sum_{r=k}^{n-1} \sum_{s=1}^{n-1} q_{r,s} \binom{s-1}{k}$.*

We conclude this section with an example of the maximal star decompositions.

Example 4. Consider again the "kite graph" G depicted in Figure 1. Let $j = 1$. The maximal j-stars associated with G are depicted in Figure 3. The j-stars corresponding to each of the 4-acyclic 4-partitions are all maximal. One of these has two edges, and the others have only one. For the j-star $\mathcal{S}$ with two edges, the corresponding complex $\mathcal{H}(\mathcal{S})$ of vector spaces is

$$0 \to \mathbb{K}^1 \xrightarrow{\Delta_2} \mathbb{K}^2 \xrightarrow{\Delta_1} \mathbb{K}^1 \to 0,$$

where

$$\Delta_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \Delta_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

The complexes corresponding to the other maximal stars are $0 \to \mathbb{K}^1 \to \mathbb{K}^1 \to 0$. Of the j-stars associated with each of the 4-acyclic 3-partitions, only four are maximal, and each of these has exactly one edge. Finally, there is exactly one maximal j-star associated with a 4-acyclic 2-partitions.

The resulting direct sum decomposition of $(\mathcal{F}_0(G))_{P_1} \otimes \kappa(P_1)$ is the following:

$$\begin{aligned} 0 \to \mathbb{K}^4 \to \mathbb{K}^9 \to \mathbb{K}^6 \to \mathbb{K}^1 \to 0 = \\ 1(0 \to \mathbb{K}^1 \to \mathbb{K}^2 \to \mathbb{K}^1 \to 0 \to 0) \\ \oplus\ 3(0 \to \mathbb{K}^1 \to \mathbb{K}^1 \to 0 \to 0 \to 0) \\ \oplus\ 4(0 \to 0 \to \mathbb{K}^1 \to \mathbb{K}^1 \to 0 \to 0) \\ \oplus\ 1(0 \to 0 \to 0 \to \mathbb{K}^1 \to \mathbb{K}^1 \to 0). \end{aligned}$$

$\square$

6. Exactness of $\mathcal{F}_1(G)$: A Gröbner Degeneration of Complexes

We know from [5] that M_G is an initial ideal of I_G. From Gröbner basis theory, there is an integral weight function that realizes this total order [7, Chapter 15]. As remarked in [13], integral weight functions that realize the degeneration from I_G to M_G arise naturally from potential theory on the graph and one such choice is the function b_q studied in [3]. In this section, we show the stronger property that the minimal free resolution of I_G also Gröbner degenerates to the minimal free resolution of M_G. As a consequence, I_G and M_G share the coarse Betti numbers. Geometrically, both I_G and M_G lie in the same subscheme of the Hilbert scheme that corresponds to varieties that share the same Betti numbers.

Let $y \in \mathrm{im}_{\mathbb{Z}^n}(\Lambda_G)$ have $y_i > 0$ for all $i \neq n$. Let the integral weight vector $\lambda \in \mathbb{Z}^n$ be a solution to the equation $\Lambda_G \lambda = y$ such that $\lambda_i > 0$ for all i from 1 to n. Note that a solution λ with these properties is guaranteed since $(1, \dots, 1)$ is in the kernel of Λ_G. For example, if we take y to be an integral multiple of the vector $(1, \dots, 1, -(n-1))$ that lies in $\mathrm{im}_{\mathbb{Z}^n}(\Lambda_G)$, the corresponding integral weight vector λ is, up to scaling, the vector b_q in [3]. An important property of the weight vector λ, as we will see in Lemma 6.1, is that the weight $\lambda \cdot D(\mathcal{C})$ of the divisor corresponding to an n-acyclic k-partition $\mathcal{C}$ is uniquely maximized among all acyclic k-partitions $\mathcal{C}'$ that are chip firing equivalent to $\mathcal{C}$, following the analogous result for acyclic orientations of G in [3, Theorem 4.14].

Given an equivalence class $\mathfrak{c}$ of acyclic k-partitions, let $\varepsilon_{\mathfrak{c}} = \lambda \cdot D(\mathcal{C})$, where $\mathcal{C} \in \mathfrak{c}$ is n-acyclic and $D(\mathcal{C})$ is viewed as an element of $\mathbb{Z}^n$. Given a monomial $m = \mathbf{x}^u e_{\mathfrak{c}} \in F_{1,k}$, we define the weight $w(m) = \lambda \cdot u + \varepsilon_{\mathfrak{c}}$. We define $\mathcal{F}_t(G)$ to be the Gröbner degeneration, i. e., the homogenization, of $\mathcal{F}_1(G)$ with respect to the integral weight function w, exactly as described in [14, Chapter 8, Section 3]. More precisely, $\mathcal{F}_t(G)$ is the $(\mathbb{Z}^n / \mathrm{im}_{\mathbb{Z}^n} \Lambda_G) \times \mathbb{Z}$-graded complex whose k^{th} homological degree is the free $R[t]$-module

$$F_{t,k} = \bigoplus_{\mathfrak{c}} R[t]\tilde{e}_{\mathfrak{c}}.$$

Here, the sum is taken over all equivalence classes $\mathfrak{c}$ of acyclic $(k+1)$-partitions, and the basis element $\tilde{e}_{\mathfrak{c}}$ has degree $(\varepsilon_{\mathfrak{c}}, w(\varepsilon_{\mathfrak{c}}))$ for any $\mathcal{C} \in \mathfrak{c}$. The k^{th} differential $\delta_{t,k} : F_{t,k} \to F_{t,k-1}$ is defined by

$$\delta_{t,k}(\tilde{e}_{\mathfrak{c}}) = t^{\varepsilon_{\mathfrak{c}}} \sum_{f} \mathrm{sign}_{\mathfrak{c}}(f) t^{-w(m_{\mathfrak{c}}(f) e_{\mathfrak{c}/f})} m_{\mathfrak{c}}(f) \tilde{e}_{\mathfrak{c}/f}. \tag{5}$$

where the sum is taken over all contractible edges f of $\mathcal{C}$. Being the homogenization of the complex $\mathcal{F}_1(G)$, the sequence $\mathcal{F}_t(G)$ is automatically a complex of $R[t]$-modules. The condition $\lambda_i > 0$ ensures a positive grading (as defined in [14, Chapter 8.3]). Note that under the evaluation map $t \mapsto 1$, the complex $\mathcal{F}_t(G)$ becomes $\mathcal{F}_1(G)$. In other words, if we consider $\mathcal{F}_t(G)$ as a family of R-complexes, then the fiber over $(t-1)$ is $\mathcal{F}_1(G)$.

Example 5. For the "kite graph" G, consider the integral weight vector $\lambda = (5, 6, 5, 2)$ with the indeterminate t having weight two. The complex $\mathcal{F}_t(G)$ is as follows:

$$0 \to R[t]^4 \xrightarrow{\delta_{t,3}} R[t]^9 \xrightarrow{\delta_{t,2}} R[t]^6 \xrightarrow{\delta_{t,1}} R[t]^1 \to 0.$$

The matrices of differentials are

$$\delta_{t,1} = \begin{pmatrix} x_1^3 - x_2x_3x_4t & x_2^2 - x_1x_4t & x_3^2 - x_1x_4t & x_1^2x_2 - x_3x_4^2t^2 & x_1^2x_3 - x_2x_4^2t^2 & x_1x_2x_3 - x_4^3t^3 \end{pmatrix}$$

$$\delta_{t,2} = \begin{pmatrix} 0 & -x_2 & -x_4t & -x_3 & -x_4t & 0 & 0 & 0 & 0 \\ -x_3^2 + x_1x_4t & -x_3x_4t & -x_1^2 & 0 & 0 & 0 & 0 & -x_4^2t^2 & -x_1x_3 \\ x_2^2 - x_1x_4t & 0 & 0 & -x_2x_4t & -x_1^2 & -x_4^2t^2 & -x_1x_2 & 0 & 0 \\ 0 & x_1 & x_2 & 0 & 0 & -x_3 & -x_4t & 0 & 0 \\ 0 & 0 & 0 & x_1 & x_3 & 0 & 0 & -x_2 & -x_4t \\ 0 & 0 & 0 & 0 & 0 & x_1 & x_3 & x_1 & x_2 \end{pmatrix}$$

$$\delta_{t,3} = \begin{pmatrix} -x_4t & 0 & 0 & x_1 \\ x_3 & 0 & -x_4t & 0 \\ 0 & x_3 & 0 & x_4t \\ -x_2 & -x_4t & 0 & 0 \\ 0 & 0 & x_2 & -x_4t \\ x_1 & x_2 & 0 & 0 \\ 0 & 0 & -x_1 & x_2 \\ -x_1 & 0 & x_3 & 0 \\ 0 & -x_1 & 0 & -x_3 \end{pmatrix}.$$

Under the grading induced by the homogenization, this complex $\mathcal{F}_t(G)$ is the minimal free resolution of the cokernel of $\delta_{t,0}$. Substituting $t = 1$ in the entries of the differentials gives the minimal resolution of I_G as shown in Example 2 and substituting $t = 0$ gives the minimal free resolution of M_G as shown in Example 3. This property holds for any graph, and we exploit these properties to prove the exactness of $\mathcal{F}_1(G)$. □

For any nonzero $t_0 \in \mathbb{K}$, the fiber $\mathcal{F}_{t_0}(G)$ over $(t - t_0)$ is isomorphic as a graded complex to $\mathcal{F}_1(G)$. In particular, the map which sends x_i to $t_0^{\lambda_i} x_i$ and $e_\mathfrak{c}$ to ${t_0}^{\varepsilon_\mathfrak{c}} e_\mathfrak{c}$ is an isomorphism. We now show that the fiber over (t) is $\mathcal{F}_0(G)$.

Lemma 6.1. *$\mathcal{F}_t(G)/t\mathcal{F}_t(G)$ is isomorphic as an R-complex to $\mathcal{F}_0(G)$.*

Proof. We must show that t divides exactly those monomials of $\delta_{t,k}(\tilde{e}_\mathfrak{c})$ corresponding to edges f of $\mathfrak{c}$ which do not appear in the n-acyclic $(k+1)$-partition $\mathcal{A} \in \mathfrak{c}$.

Let f be a contractible edge of $\mathfrak{c}$, and let $\mathcal{C} \in \mathfrak{c}/f$ be n-acyclic. Let $\hat{\mathcal{C}} \in \mathfrak{c}$ be the acyclic k-partition obtained from $\mathcal{C}$ by introducing the edge f (with its given orientation), as explained in Remark 2. Note that $\hat{\mathcal{C}}$ is not in general n-acyclic. Now by definition, $m_\mathfrak{c}(f) = \mathbf{x}^{D(\hat{\mathcal{C}}) - D(\mathcal{C})}$, and it follows that $w(m_\mathfrak{c}(f)e_{\mathfrak{c}/f}) = \lambda \cdot D(\hat{\mathcal{C}})$.

Thus, it suffices to show that if $\mathcal{A} \in \mathfrak{c}$ is n-acyclic and $\mathcal{B} \in \mathfrak{c}$ is not, then $\lambda \cdot D(\mathcal{A}) > \lambda \cdot D(\mathcal{B})$. Note that the divisor $D(\mathcal{B})$ is obtained from $D(\mathcal{A})$ by a sequence of vertex firings not including n (c. f. Remark 1). Thus, there is some nonzero $\sigma \in \mathbb{N}^n$ with $\sigma_n = 0$ such that $\sigma\Lambda_G = D(\mathcal{A}) - D(\mathcal{B})$. Hence,

$$\lambda \cdot (D(\mathcal{A}) - D(\mathcal{B})) = \sigma^T \Lambda_G \lambda = \sigma^T \cdot y > 0$$

as required. □

Remark 3. Note that Lemma 6.1 crucially uses the property that the weight of an n-acyclic k-partition $\mathcal{C}$ is uniquely maximized among acyclic k-partitions $\mathcal{C}'$ which are chip firing equivalent to $\mathcal{C}$.

As a result of Lemma 6.1, it follows that $\mathcal{F}_t(G)$ is exact.

Theorem 6.2. *Let I_G^t denote the homogenization of I_G with respect to the weight function w. The sequence*

$$\mathcal{F}_t(G) : 0 \to F_{t,n-1} \xrightarrow{\delta_{t,n-1}} \cdots \xrightarrow{\delta_{t,2}} F_{t,1} \xrightarrow{\delta_{t,1}} F_{t,0}$$

defined by Equation (5) *is a minimal free resolution of R/I_G^t.*

Proof. We have already noted that $\mathcal{F}_t(G)$ is a complex. We now show that it is exact.

Observe that $H_k(\mathcal{F}_t(G)) \otimes R[t]/(t)$ includes into $H_k(\mathcal{F}_t(G) \otimes R[t]/(t)) = H_k(\mathcal{F}_0(G))$. Since the latter is trivial by Lemma 6.1 and Theorem 4.1, so is the former. Consider the short exact sequence

$$0 \to \mathbb{K}[t] \xrightarrow{t} \mathbb{K}[t] \to \mathbb{K}[t]/(t) \to 0,$$

and tensor it over $R[t]$ with $H_k(\mathcal{F}_t(G))$. Since $- \otimes H_k(\mathcal{F}_t(G))$ is right-exact, it follows that multiplication by t is a surjection from $H_k(\mathcal{F}_t(G))$ onto itself. By Nakayama's lemma, it follows that $H_k(\mathcal{F}_t(G))$ is trivial, i. e., $\mathcal{F}_t(G)$ is exact.

As we observed in the proof of Lemma 6.1, firing any set of vertices not including n produces a divisor with smaller weight with respect to λ. It then follows from [5, Theorem 14] that the set

$$\Gamma = \{\delta_{t,1}(\mathfrak{c}) : \mathfrak{c} \text{ a chip-firing equivalence class of acyclic 2-partitions}\}$$

is a Gröbner basis for I_G under a monomial order respecting the weight function w. Since the homogenization of a Gröbner basis is a Gröbner basis for the homogenization of an ideal, it follows that the homogenization of I_G is the image of $\delta_{t,1}$, and so $\mathcal{F}_t(G)$ is a free resolution for R/I_G^t. Finally, we note that $\mathcal{F}_t(G)$ is minimal since the image of $\delta_{t,k}$ is contained in $\langle x_1, \dots, x_n \rangle \cdot F_{t,k-1}$ for all k from 1 to $n-1$. □

We now arrive at the crucial property of Gröbner degeneration, from which the exactness of $\mathcal{F}_1(G)$ will follow.

Proposition 6.3 ([14, Proposition 8.26])**.** *Let M be a graded submodule of $F_{1,k}(G)$ for some integer k, and denote by $\tilde{M}$ its homogenization with respect to the weight function w. Then $F_{t,k}(G)/\tilde{M}$ is free as a $\mathbb{K}[t]$-module.*

We are finally able to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2, we know that $\mathcal{F}_1(G)$ is a complex such that the cokernel of $\delta_{1,1}$ is R/I_G. Since $t-1$ is not a zero-divisor for R/I_G^t by Proposition 6.3, the exactness of $\mathcal{F}_1(G)$ follows from the exactness of $\mathcal{F}_t(G)$ (see [14, Proposition 8.28]). As with $\mathcal{F}_t(G)$ and $\mathcal{F}_0(G)$, the minimality of $\mathcal{F}_1(G)$ is clear: the image of $\delta_{1,k}$ is contained in $\langle x_1, \dots, x_n \rangle \cdot F_{1,k-1}$ for all k from 1 to $n-1$.

□

Corollary 6.4. *The minimal free resolution of M_G is a Gröbner degeneration of the minimal free resolution of I_G.*

Remark 4. If we homogenize the minimal free resolution of an arbitrary $\mathbb{Z}$-graded ideal I then the fiber of the resulting complex at 0 will in general not be exact. Instead, the well known upper semicontinuity holds [14, Theorem 8.29]: the Betti numbers of the ideal I is at most the corresponding Betti numbers of any initial ideal of I.

7. CW complex supporting $\mathcal{F}_0(G)$

A free resolution of a monomial ideal is supported by a CW complex if the differentials of the free resolution are given by an appropriate modification of the differentials of the CW complex (see [14]). In general, minimal free resolutions of monomial ideals are not supported by CW complexes [19]. However, in this section we will show that $\mathcal{F}_0(G)$ is supported on a CW complex with a fairly simple structure.

Theorem 7.1. *The complex $\mathcal{F}_0(G)$ is a cellular resolution, i. e., it is supported on a CW complex.*

The CW complex supporting $\mathcal{F}_0(G)$ has k-cells corresponding to the n-acyclic $(k+2)$-partitions of G and has the same poset structure as the n-acyclic partitions under the refinement ordering. In the remainder of this section, we show that this CW complex is well-defined, and then conclude that Theorem 7.1 follows.

Given a graph G on $[n]$, we recursively define an associated cell complex $\mathrm{Part}(G)$ as follows. We introduce a 0-cell $e_{\mathcal{C}}$ for each n-acyclic 2-partition $\mathcal{C}$. When the $(k-1)$-skeleton of $\mathrm{Part}(G)$ is defined, and $\mathcal{C}$ is an n-acyclic $(k+2)$-partition, define

$$\partial e_{\mathcal{C}} = \cup_f \overline{e_{\mathcal{C}/f}}.$$

where the union is taken over all contractible edges f of $\mathcal{C}$. Proposition 7.3 below shows that $\partial e_{\mathcal{C}} \cong S^{k-1}$. We introduce a k-cell $e_{\mathcal{C}}$ for each n-acyclic $(k+2)$-partition $\mathcal{C}$ by gluing it to $\partial e_{\mathcal{C}}$ along a homeomorphism with the sphere. Thus, the closure of each k-cell in $\mathrm{Part}(G)$ is homeomorphic to the k-disk D^k.

For any $\overline{e_{\mathcal{C}}}, \overline{e_{\mathcal{C}'}} \subset \mathrm{Part}(G)$, we have $\overline{e_{\mathcal{C}}} \cap \overline{e_{\mathcal{C}'}} = \overline{e_{[\mathcal{C},\mathcal{C}']}}$, where $[\mathcal{C},\mathcal{C}']$ is the finest n-acyclic partition refined by both $\mathcal{C}$ and $\mathcal{C}'$. This oriented partition $[\mathcal{C},\mathcal{C}']$ is well-defined, and is given by taking the finest partition refined by $\Pi(\mathcal{C})$ and $\Pi(\mathcal{C}')$, contracting all edges on which the orientations of $\mathcal{C}$ and $\mathcal{C}'$ do not agree, and then iteratively contracting all cycles among the remaining oriented edges.

Definition 7.2. If $\mathcal{C}$ is an n-acyclic k-partition of G, and A is a set of edges appearing in $\mathcal{C}$, we say that A is mutually contractible if any subset of A can be contracted without creating any cycles.

Proposition 7.3. *Let $k > 0$. For any n-acyclic $(k+2)$-partition $\mathcal{C}$, the subcomplex $\partial e_{\mathcal{C}}$ of $\mathrm{Part}(G)$ is homeomorphic to S^{k-1}.*

Proof. The claim holds for the two connected graphs on three vertices. We now assume the claim is true for graphs on at most $n-1$ vertices and let G have n vertices. For $k+2 < n$, the claim holds since for n-acyclic $(k+2)$-partitions $\mathcal{C}$, we have $\partial e_{\mathcal{C}}$ homeomorphic to the corresponding subcomplex of $\mathrm{Part}(G_{\Pi(\mathcal{C})})$, and in particular the $(n-3)$-skeleton Z of $\mathrm{Part}(G)$ is well-defined. Fixing an n-acyclic n-partition $\mathcal{C}$, it remains to show that $\partial e_{\mathcal{C}} \cong S^{n-3}$.

Note that if $e_{\mathcal{C}'} \subset \partial e_{\mathcal{C}}$ is an $(n-3)$-cell, then $\mathcal{C}'$ is obtained from $\mathcal{C}$ by contracting a unique edge. Furthermore, by the inductive hypothesis, $\overline{e'_{\mathcal{C}}} \cong D^k$ for any n-acyclic $(k+2)$-partition $\mathcal{C}'$ with $k+2 < n$. For any nonempty mutually contractible set of edges $A \subset \mathcal{C}$, define $\mathcal{C}_A$ as the oriented partition given by contracting A, let $D_A = \overline{e_{\mathcal{C}_A}}$, and let

$$X_A = \cup_{e \in A} D_{\{e\}} \subset \partial e_{\mathcal{C}}$$

Now let $A \subset \mathcal{C}$ be mutually contractible, and note that if $B \subset A$ has $|B| = r > 0$ then $D_B \cong D^{n-(r+2)}$. Furthermore, for any nonempty $B_1, B_2 \subset A$, we have

$D_{B_1} \cap D_{B_2} = D_{B_1 \cup B_2}$. It follows that if there exists an edge $(u, v) \in A$ for every nonsink vertex u, then X_A has the cell structure of a hollow simplex, so $X_A \cong S^{n-3}$. Otherwise, $X_A \cong D^{n-3}$.

Note that if G is a tree, then the set A of all edges in $\mathcal{C}$ is mutually contractible, and thus $\partial e_{\mathcal{C}} = X_A \cong S^{n-3}$ by the previous paragraph. Thus, we proceed by induction on the number of edges of G. Let $u \in V$ have at least two out-neighbors in $\mathcal{C}$ and be such that for any v with a directed path to u, there is a unique out-neighbor of v. Clearly such a vertex u exists whenever G is not a tree because $\mathcal{C}$ is acyclic. An edge of the form $e = (u, v)$ can be legally contracted if and only if every directed path from u to v contains (u, v). If $e = (u, v)$ cannot be legally contracted, then let $\mathcal{C}'$ be the n-acyclic n-partition of $G \setminus \{e\}$ which agrees with $\mathcal{C}$ on all edges other than e. Then $\partial e_{\mathcal{C}}$ is homeomorphic to $\partial e_{\mathcal{C}} \subset \mathrm{Part}(G \setminus \{e\})$.

Thus, without loss of generality, every out-edge of u can be legally contracted. In fact the set A of out-edges of u is mutually contractible, since contracting any subset of A does not create any paths between the out-neighbors of u.

Claim. For any disk $\overline{e'_{\mathcal{C}}} \in \partial e_{\mathcal{C}} \setminus \cup_{f \in A} e_{\mathcal{C}_f}$ with $\mathcal{C}'$ an n-acyclic $(k+2)$-partition of G, we have $\overline{e'_{\mathcal{C}}} \cap X_A \cong D^{k-1}$.

Let $\mathcal{C}'$ be as in the claim, and let $B \subset A$ be the collection of edges (u, v) that cannot be legally contracted from $\mathcal{C}'$. We have $\partial e_{\mathcal{C}'}$ homeomorphic to the corresponding subcomplex of $\mathrm{Part}(G'_{\mathcal{C}})$, which is homeomorphic to the corresponding subcomplex of $\mathrm{Part}(G'_{\mathcal{C}} \setminus B)$. In the latter graph, $A \setminus B$ is mutually contractible, and $X_{A \setminus B} \cong D^{k-1}$. The claim follows.

We now show that $\partial e_{\mathcal{C}} \cong S^{n-3}$ follows from the claim. Let $f \in A$, let $G' = G \setminus (A \setminus \{f\})$, and let $\mathcal{C}'$ be the n-acyclic n-partition of G' corresponding to $\mathcal{C}$. We have $\partial e_{\mathcal{C}'} \cong S^{n-3}$ by the inductive hypothesis since $|A| \geq 2$. On the other hand, note that for any $\mathcal{A}$ refined by $\mathcal{C}'$, if f appears in $\mathcal{A}$ then f is legally contractible in $\mathcal{A}$. This follows from the fact that u has a unique out-neighbor in $\mathcal{C}'$ and the induced subgraph on all vertices with a path to u is a tree. Thus, for any $\overline{e_{\mathcal{A}}} \subset \partial e_{\mathcal{C}'} \setminus e_{\mathcal{C}'_f}$ with $\mathcal{A}$ an n-acyclic $(k+2)$-partition of G', we have $D_f \cap \overline{e_{\mathcal{A}}} \cong D^{k-1}$. On the other hand, if $\mathcal{A}$ and $\mathcal{B}$ are refined by $\mathcal{C}$ and do not join u to any of its out neighbors, then the corresponding acyclic partitions of G' are both refined by $\mathcal{C}'$ as well, and $[\mathcal{A}, \mathcal{B}]$ is the same in G as in G'. It then follows from the claim that we have a homeomorphism $\partial e_{\mathcal{C}} \cong \partial e_{\mathcal{C}'}$ given by mapping X_A to D_f and preserving the other cells. $\square$

We remark that when G is saturated, each n-acyclic k-partition $\mathcal{C}$ has exactly $k-1$ contractible edges, which are in fact mutually contractible. In that case, $\mathrm{Part}(G)$ has the cell structure of a simplicial complex, and in fact it is the Scarf complex described by Postnikov and Shapiro in [17].

Proof of Theorem 7.1. We show that after appropriately labeling the cells of $\mathrm{Part}(G)$ with monomials in R, the resulting cellular complex is isomorphic to $\mathcal{F}_0(G)$ (cf. [14, Chapter 4]).

Label each cell $e_{\mathcal{C}}$ of $\mathrm{Part}(G)$ with the monomial $\mathbf{x}^{D(\mathcal{C})}$. We claim that the label of $e_{\mathcal{C}}$ is the least common multiple of the labels of the cells in $\partial e_{\mathcal{C}}$, so that $\mathrm{Part}(G)$ is a *labeled CW complex.* Certainly $\mathbf{x}^{D(\mathcal{C}/e)}$ divides $\mathbf{x}^{D(\mathcal{C})}$ for every contractible edge e of $\mathcal{C}$. On the other hand, if $\mathcal{C}$ is an n-acyclic k-partition for some $k \geq 3$, then $\mathcal{C}$ then for every vertex j of G, there is a contractible edge e of $\mathcal{C}$ such that

$j \notin e^-$. Then $(D(\mathcal{C}/e))_j = (D(\mathcal{C}))_j$, and the claim follows. Identifying the standard basis elements of the cellular complex for $\mathrm{Part}(G)$ with the standard basis elements of $\mathcal{F}_0(G)$, we see that the cellular monomial matrices associated with $\mathrm{Part}(G)$ are exactly the differential maps of $\mathcal{F}_0(G)$, up to signs. Since the cellular monomial matrices associated with $\mathrm{Part}(G)$ form a complex, if we define the sign function $\mathrm{sign}_{\mathfrak{c}}$ of $\mathcal{F}_0(G)$ to be the sign of the corresponding entry from the cellular complex for $\mathrm{Part}(G)$, we see that $\mathrm{sign}_{\mathfrak{c}}$ satisfies the required Property (1). Thus, under an appropriate sign function, $\mathcal{F}_0(G)$ is supported on $\mathrm{Part}(G)$. □

8. Conclusion

In this paper, we explicitly constructed minimal free resolutions of the ideals M_G and I_G. The general version of this approach is to associate a combinatorial object (such as a graph or a simplicial complex) with a graded module and attempt to describe the minimal free resolution of the graded module in terms of the underlying combinatorial structure. It is natural to ask whether directed graphs are suitable for this purpose, i. e., whether the techniques of this paper generalize to toppling ideals and G-parking function ideals of directed graphs G. As we noted in the introduction, any lattice ideal coming from a full rank submodule of the root lattice can be realized as the Laplacian lattice ideal of a directed graph. Thus, the question of finding minimal free resolutions of such lattice ideals is both challenging and exciting.

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